

Chern-Simons action for inhomogeneous Virasoro group as an extension of three dimensional flat gravity

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ABSTRACT. We initiate the study of a Chern-Simons action associated to the semi-direct sum of the Virasoro algebra with its coadjoint representation. This model extends the standard Chern-Simons formulation of three dimensional flat gravity and is similar to the higher-spin extension of three dimensional anti-de Sitter or flat gravity. The extension can also be constructed for the exotic but not for the cosmological constant deformation of flat gravity.

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1 Introduction

Higher spin extensions of AdS_3 gravity [1] have attracted a lot of attention recently (see e.g. [2, 3, 4, 5, 6]) as they allow one to probe novel aspects of the AdS/CFT correspondence beyond the purely gravitational sector in the most tractable setting of three bulk dimensions. More recently, such higher spin extensions have also been constructed for three dimensional flat gravity [7, 8].

A key feature of the Chern-Simons formulation of pure gravity in flat space [9, 10] is the fact that the action does not involve the Killing metric like it does for semi-simple Lie algebras, but the non-degenerate invariant pairing that exists between the translation and the rotation generators in three dimensions. More generally, such a metric, and thus also a Chern-Simons model, exists for all algebras $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$ that are the semi-direct sum of a given algebra \mathfrak{g} with its coadjoint representation \mathfrak{g}^* embedded as an abelian ideal [11, 12].

Since the Lorentz algebra $\mathfrak{so}(2, 1)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, probably the most obvious infinite-dimensional extension of flat gravity consists in replacing $\mathfrak{sl}(2, \mathbb{R})$ by the Virasoro algebra vir . Such a Chern-Simons Virasoro model involves all ingredients that are used when applying the orbit method to the study of representations of the Virasoro algebra (see e.g. [13, 14, 15, 16] for considerations in the physics literature). One might hope that the model can be useful in this context, in the same way as the Poisson sigma model [17, 18] is relevant to the problem of quantizing Poisson manifolds [19, 20].

A further motivation to study the model comes from the representation theory of the symmetry algebra of asymptotically flat three dimensional spacetimes [21] (see also [22]). This algebra is given by the BMS algebra in three dimensions, which is the semi-direct sum of the algebra of vector fields on the circle with its adjoint representation embedded as an abelian ideal, $\mathfrak{bms}_3 = \text{Vect}(S^1) \ltimes_{\text{ad}} \text{Vect}(S^1)_{\text{ab}}$. Because of theorems available on unitary irreducible representations of finite dimensional Lie algebras and groups with this particular structure (see e.g. [23]), it is natural to study induced representations, for which it is again the coadjoint representation of the Virasoro algebra with its orbits and little groups that is relevant [24]. Finally, the solution space of three-dimensional asymptotically anti-de Sitter spacetimes is also classified by (two copies of) the coadjoint orbits of the Virasoro group (see e.g. [25, 26] for recent discussions). In the flat case, the coadjoint orbits of the BMS_3 group are controlled by Virasoro coadjoint orbits in a standard way appropriate to semi-direct products [27].

After reviewing the formulation of Chern-Simons theories for inhomogeneous groups, we start our analysis of the model with a detailed discussion of how flat gravity and its solutions are included in this infinite-dimensional extension. We will show that gravity solutions can be included in the extended model at the price of turning on additional non-gravity modes that, however, do not backreact on the geometry.

We then study its consistent deformations. This is most conveniently done in the context of the Batalin-Vilkovisky antifield formalism [28, 29, 30, 31, 32] (see also [33, 34] for reviews) where they are controlled by the cohomology of the BV differential in the space of local functionals in ghost number 0 [35] (see also [36]). In particular, in the case of $\mathfrak{iso}(2, 1)$, the deformations provided in [10] have been studied from this point of view in [37], section 7. More generally, in any ghost number, the BV cohomology in the space of local functionals is locally isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra with which the Chern-Simons theory is constructed. This is a particular case of a general result valid for AKSZ sigma models [38] for which the cohomology of the BV differential in the space of local functionals is locally isomorphic to the cohomology of the target space differential [39]. An interesting feature is that the other primitive generator in degree 3 of the cohomology ring of $\text{Vect}(S^1)$, appears in this deformation.

We end with considerations on the dual boundary theory. Chern-Simons theories on a solid cylinder induce chiral Wess-Zumino-Witten theories on their boundary [40, 41]. For the Chern-Simons formulation of three dimensional gravity, the boundary theory has been studied in the flat case in [42], in the anti-de Sitter case with non trivial asymptotics in [43], and in the flat case with non trivial asymptotics at null infinity in [44]. For the Chern-Simons Virasoro model, one ends up with a chiral Wess-Zumino model with a current algebra determined by $\mathfrak{vir} \ltimes_{\text{ad}^*} \mathfrak{vir}^*$. A classical Sugawara construction then manifestly shows that the model is conformally invariant.

2 Chern-Simons model for inhomogeneous groups

The Chern-Simons formulation of three dimensional flat gravity is based on the Poincaré algebra $\mathfrak{iso}(2, 1)$. If the generators are denoted by P_a, J_b , the invariant inner product required to construct the action is given by $\langle P_a, J_b \rangle = \eta_{ab}$, $\langle P_a, P_b \rangle = 0 = \langle J_a, J_b \rangle$ with η_{ab} the Minkowski metric in three dimensions. This construction can be generalized by replacing $\mathfrak{so}(2, 1)$ with a generic Lie algebra \mathfrak{g} : the total Lie algebra is the semi-direct sum of \mathfrak{g} , whose generators are denoted by e_a , with its coadjoint representation embedded as an abelian ideal,

$$[e_a, e_b] = f_{ab}^c e_c, \quad [e_a, e^{*b}] = -f_{ac}^b e^{*c}, \quad [e^{*a}, e^{*b}] = 0. \quad (2.1)$$

The reason why it is always possible to construct a Chern-Simons action for such a Lie algebra is the existence of the non-degenerate invariant inner product

$$\langle e_a, e^{*b} \rangle = \delta_a^b, \quad \langle e_a, e_b \rangle = 0 = \langle e^{*a}, e^{*b} \rangle. \quad (2.2)$$

3 Chern-Simons Virasoro model

What we want to do here is use as Lie algebra \mathfrak{g} the Virasoro algebra \mathfrak{vir} with generators L_m, Z ,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{Z}{12}\delta_{m+n}^0 m(m^2 - 1), \quad [L_m, Z] = 0, \quad (3.1)$$

so that

$$\begin{aligned} [L_m, L^{*n}] &= (n - 2m)L^{*n-m}, \quad [L_m, Z^*] = -\frac{1}{12}m(m^2 - 1)L^{*-m}, \\ [Z, L^{*n}] &= 0 = [Z, Z^*], \quad [L^{*m}, L^{*n}] = 0 = [L^{*m}, Z^*]. \end{aligned} \quad (3.2)$$

If the associated gauge field is denoted by $A = A_\mu^A T_A dx^\mu$, with x^μ local coordinates on a three-dimensional manifold \mathcal{M}_3 and $T_A = (L_m, Z, L^{*m}, Z^*)$, the Chern-Simons action is

$$S_0[A] = \kappa \int_{\mathcal{M}_3} \frac{1}{2} \langle A, dA + \frac{2}{3}A^2 \rangle. \quad (3.3)$$

In more details, let $A = A^m L_m + DZ + B_m L^{*m} + CZ^*$. The associated curvatures are

$$\begin{aligned} F^m &= dA^m + \frac{1}{2}f_{nk}^m A^n A^k, \quad F^Z = dD + \frac{1}{24}m(m^2 - 1)A^{-m}A^m, \\ F_m &= dB_m - f_{nm}^k A^n B_k + \frac{1}{12}m(m^2 - 1)A^{-m}C, \quad F_{Z^*} = dC, \end{aligned} \quad (3.4)$$

where $f_{mn}^k = (m - n)\delta_{m+n}^k$. The equations of motions are equivalent to requiring these curvatures to vanish.

This theory can also be understood as a three dimensional BF-type theory for the Virasoro algebra. Indeed, through integrations by parts, action (3.3) can be rewritten as

$$S_0 = \kappa \int_{\mathcal{M}_3} (B_m F^m + C F^Z). \quad (3.5)$$

Finally, introducing an additional circle S^1 with coordinate $\phi \in [0, 2\pi)$, elements of \mathfrak{vir} are pairs $(v, -ia)$ with $v = f(\phi)\partial_\phi$ vector fields on the circle and a a real number. In particular, $L_m = (ie^{im\phi}\partial_\phi, 0)$ for $m \neq 0$, $L_0 = (i\partial_\phi, \frac{1}{24})$, $Z = (0, 1)$. The commutation relations are given by

$$[(v_1, -ia_1), (v_2, -ia_2)] = \left((f_1 f_2' - f_2 f_1') \partial_\phi, -\frac{i}{48\pi} \int_0^{2\pi} d\phi (f_1' f_2'' - f_2' f_1'') \right). \quad (3.6)$$

Coadjoint vectors are given by pairs (u, it) , where $u = h(\phi)d\phi^2$ is a quadratic differential and t a real number. In these terms, the invariant inner product is given by

$$\langle (u, it), (v, -ia) \rangle = \int_0^{2\pi} d\phi h f + at, \quad (3.7)$$

with coadjoint action

$$ad_{(f, -ia)}^*(u, it) = \left((fh' + 2f'h - \frac{t}{24\pi} f''') d\phi^2, 0 \right). \quad (3.8)$$

In particular, $L^{*m} = (-\frac{ie^{-im\phi}}{2\pi}, 0)$, $Z^* = (\frac{i}{48\pi}, 1)$.

Let $(\mathcal{A}, -iD)$ and (\mathcal{B}, iC) be one forms on \mathcal{M}_3 with values in \mathfrak{vir} respectively \mathfrak{vir}^* and let $(\mathcal{F}, -iF^Z) = d(\mathcal{A}, -iD) + \frac{1}{2}[(\mathcal{A}, -iD), (\mathcal{A}, -iD)]$. Instead of an infinite sum over m , the Chern-Simons action (3.5) can then be written with an additional integral over ϕ as

$$S_0 = \kappa \int_{\mathcal{M}_3} \langle (\mathcal{B}, iC), (\mathcal{F}, -iF^Z) \rangle, \quad (3.9)$$

and interpreted as a field theory on $\mathcal{M}_3 \times S^1$.

Interestingly, the construction of Chern-Simons theories for inhomogeneous groups as described in section 3 can also be done by using as \mathfrak{g} the semi-direct sum of the Virasoro algebra with the affine Kac-Moody algebra for $\mathfrak{sl}(2, \mathbb{R})$. A Chern-Simons theory of precisely this type (but without the Virasoro central extension) has appeared previously in the context of a Kaluza-Klein reduction for four dimensional gravity on a circle in [45, 46].

4 Extension of flat gravity

The Virasoro Chern-Simons model may be interpreted as describing the non trivial coupling of an infinite number of additional gauge fields to gravity in three dimensions since

the action reduces to the Einstein-Hilbert one by putting to zero the gauge fields associated with the generators Z, Z^* and L_m, L^{*m} for $m \neq -1, 0, 1$. Indeed, when all gauge fields besides $A^m, B_m, m, = -1, 0, 1$ are switched off, we get

$$S_0 = \kappa \int B_m F^m, \quad m, n, k \in -1, 0, 1. \quad (4.1)$$

The standard gravitational formulation is recovered through a change of basis: $A^m L_m + B_m L^{*m} = \omega^a J_a + e^a P_a$, with $a, b = 0, 1, 2$, $J^a = -\frac{1}{2} \epsilon^{abc} J_{bc}$, $[J_a, J_b] = \epsilon_{abc} J^c$, $[J_a, P_b] = \epsilon_{abc} P^c$, $[P_a, P_b] = 0$, $\epsilon_{012} = 1$, where ω^a and e^a are the spin connection and vielbein 1-forms, respectively. Indices are lowered and raised with $\eta_{ab} = \text{diag}(-1, 1, 1)$ and its inverse while

$$\begin{aligned} l_{-1} &= J_0 - J_1, & l_1 &= J_0 + J_1, & l_0 &= -J_2, \\ P_0 &= -l^{*1} - l^{*-1}, & P_1 &= l^{*1} - l^{*-1}, & P_2 &= -l^{*0}. \end{aligned} \quad (4.2)$$

and action (4.1) coincides with the Einstein-Hilbert action in terms of dreibeins and spin connections,

$$S_0[e, \omega] = \frac{1}{16\pi G} \int e e_a^\mu e_b^\nu R_{\mu\nu}^{ab} d^3x, \quad (4.3)$$

provided $\kappa = -\frac{1}{8\pi G}$.

The question is then whether any solution to the gravity equations of motions can be lifted to a solution of the extended system. More generally, consider a Chern-Simons theory based on $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. In the gravity case, $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R})$, while $\mathfrak{g} = \mathfrak{vir}$. Can any solution of the Chern-Simons theory based on $\mathfrak{h} \ltimes_{\text{ad}^*} \mathfrak{h}^*$ be lifted to a solution of the Chern-Simons theory based on $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$?

Answering this question is not completely trivial because $\mathfrak{h} \ltimes_{\text{ad}^*} \mathfrak{h}^*$ is not a subalgebra of $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$. Indeed, let us denote by e_γ generators of \mathfrak{h} and by $e_\Gamma = (e_\gamma, e_C)$ generators of \mathfrak{g} . The equations of motion of the theory based on \mathfrak{g} are

$$dA^\Gamma + \frac{1}{2} f_{\Delta\Sigma}^\Gamma A^\Delta A^\Sigma = 0, \quad dB_\Gamma - f_{\Sigma\Gamma}^\Delta A^\Sigma B_\Delta = 0, \quad (4.4)$$

while those of the theory based on \mathfrak{h} are similar but with upper case Greek indices replaced by lower case ones. Now, while it is true that given a solution of the theory based on \mathfrak{h} , the first set of equations in (4.4) can always be solved by setting to zero the all gauge fields A^C since $f_{\gamma\delta}^S = 0$, this cannot be done for the second set of equations since switching off all fields B_C results in the additional constraints $f_{\delta C}^\gamma A^\delta B_\gamma = 0$ on the fields of the theory based on \mathfrak{h} .

Let us then provide a formal argument showing that every solution of the theory based on \mathfrak{h} can be lifted to a solution of the theory based on \mathfrak{g} by keeping the fields B_γ unchanged and suitably turning on additional fields B_C . Indeed, locally, the general solution of the theory based on \mathfrak{h} is given by

$$A = h^{-1} dh, \quad B = h^{-1} (dC_\gamma e^{*\gamma}) h, \quad (4.5)$$

with h a group element associated to \mathfrak{h} and $C_\gamma(x)$ arbitrary spacetime-dependent functions. The lift of the solution is then simply obtained by considering the formal group element $h = e^{k^\alpha e_\alpha}$ with spacetime dependent functions $k^\alpha(x)$ to be an element of the group associated to \mathfrak{g} . This gives

$$B = dC_\gamma e^{-[k^\alpha e_\alpha]} e^{*\gamma} = dC_\gamma (e^{*\gamma} + k^\alpha f_{\alpha\Delta}^\gamma e^{*\Delta} + \frac{1}{2} k^\alpha k^\beta f_{\beta\Sigma}^\gamma f_{\alpha\Delta}^\Sigma e^{*\Delta} + \dots). \quad (4.6)$$

The reason why B_γ is unchanged is that

$$[\mathfrak{h}, [\mathfrak{h}, \dots [\mathfrak{h}, \mathfrak{g}^*] \dots]]|_{h^*} = [\mathfrak{h}, [\mathfrak{h}, \dots [\mathfrak{h}, \mathfrak{h}^*] \dots]]|_{\mathfrak{h} \times_{\text{ad}^*} \mathfrak{h}^*}$$

since \mathfrak{h} is a subalgebra of \mathfrak{g} .

Note however that the exponential map does not define a local chart close to the identity for the diffeomorphism of the circle or the Virasoro group. The above argument then only makes sense if one thinks about these groups formally, as done in the current context for instance in [47, 15].

More concretely, let us consider for instance the general solution of three-dimensional flat gravity in BMS gauge [48]. The associated vielbein and spin connections are given in [44]. Explicitly, when taking into account (4.2), they translate to

$$\begin{aligned} A^{\pm 1} &= \left(\frac{M}{2} \mp 1\right) d\phi, \quad A^0 = 0, \\ B_{\pm 1} &= -\frac{1}{2} \left(\left(\frac{M}{2} \pm 1\right) du - dr + Nd\phi\right), \quad B_0 = -rd\phi. \end{aligned} \quad (4.7)$$

Here M and N are defined in terms of two arbitrary function of ϕ , $\Theta(\phi)$ and $\Xi(\phi)$ through $M = \Theta$, $N = \Xi + \frac{u}{2}\Theta'$.

According to the previous considerations, we can assume $A^m = 0 = D$ for $|m| > 1$. Let us then show that it is enough to turn on in addition $B_{\pm 2}$ in order to extend the above solutions.

Indeed, if in addition $B_m = 0 = C$ for $|m| > 2$, the equations for dB_m, dC are automatically fulfilled for $|m| > 3$. The equations for $|m| = 2, 3$ read explicitly

$$\begin{aligned} dB_2 + 3A^{-1}B_1 + A^1B_3 + 2A^0B_2 &= 0, \\ dB_{-2} - 3A^1B_{-1} - A^{-1}B_{-3} - 2A^0B_{-2} &= 0, \\ A^{-1}B_2 = 0, \quad A^1B_{-2} &= 0. \end{aligned} \quad (4.8)$$

Choosing $B_2 = f_+ A^{-1}$ and $B_{-2} = f_- A^1$ for some functions f_+ and f_- allows one to satisfy the last 2 equations, while the first 2 turn into a system of partial differential equations for f_+ and f_- ,

$$d(f_+ A^{-1}) + 3A^{-1}B_1 + 2f_+ A^0 A^1 = 0, \quad (4.9)$$

$$d(f_- A^1) - 3A^1B_{-1} - 2f_- A^0 A^{-1} = 0. \quad (4.10)$$

By using that for (4.7), $A_0 = 0, dA^{\pm 1} = 0$, they reduce to $df_{\pm} = \mp 3B_{\pm 1}$. The associated integrability conditions $dB^{\pm 1} = 0$ are satisfied for (4.7) and

$$f_{\pm} = \pm \frac{3}{2} \left(\frac{u}{2} \Theta \pm u + \tilde{\Xi} - r \right), \quad (4.11)$$

where $\tilde{\Xi}' = \Xi$.

Then, we see explicitly that the field equations for the gravity modes are indeed satisfied. The gravity modes, even when they source the extra modes with $|m| > 1$, do it in such a way that the latter, if originally switched off, do not backreact on Einstein's field equations.

5 Deformations of the Chern-Simons Virasoro model

Studying consistent deformations of the Chern-Simons action (3.3) in the context of the Batalin-Vilkovisky antifield formalism involves several steps.

(i) All infinitesimal deformations must belong to the cohomology group $H^{0,3}(s|d)$, where s is the antifield dependent BRST differential associated with the Batalin-Vilkovisky master action for the Virasoro Chern-Simons action (3.3).

(ii) For Chern-Simons theories, $H^{0,3}(s|d)$ is isomorphic to the Chevalley-Eilenberg cohomology of the associated Lie algebra.

For simplicity, let us first restrict ourselves to the non-centrally extended case of the Lie algebra $\mathfrak{witt} \ltimes \mathfrak{witt}^*$ with generators l_m, l^{*n} , so that $c, Z, Z^* = 0$. In this case, $H^{0,3}(s|d) \cong H^3(\mathfrak{witt} \ltimes \mathfrak{witt}^*)$.

(iii) The Hochschild-Serre analysis [49] can be adapted to the current problem. Denoting the generators of $\wedge(\mathfrak{witt} \ltimes \mathfrak{witt}^*)^*$ by η^m and C_m , the Chevalley-Eilenberg coboundary operator can be written as

$$\gamma = -\frac{1}{2} \eta^m \eta^n f_{mn}^k \frac{\partial}{\partial \eta^k} + \eta^m \rho(l_m), \quad \rho(l_m) = f_{mn}^k C_k \frac{\partial}{\partial C_n}. \quad (5.1)$$

Let $N_{\eta} = \eta^m \frac{\partial}{\partial \eta^m}$ and $N_C = C_m \frac{\partial}{\partial C_m}$. Since $[\gamma, N_C] = 0$, the problem decomposes into separate cohomology problems with eigenvalues $(3, 0), (2, 1), (1, 2), (0, 3)$ for (N_{η}, N_C) .

In degree $(3, 0)$ there is but one cohomology class ([50], [51], Theorem 2.4.2), which can be written as $\beta = \frac{1}{3!} \beta_{mnk} \eta^m \eta^n \eta^k$ with

$$\beta_{mnk} = \beta_{[mnk]} = \delta_{m+n+k}^0 (mn(m-n) + nk(n-k) + mk(k-m)). \quad (5.2)$$

If the gauge field is denoted by $B^m l_m + A_m l^{*m}$, the BV master action is given by

$$S = \kappa \int [A_m dB^m + \frac{1}{2} A_k f_{mn}^k B^m B^n + \star A^{*m} (dC_m + f_{mn}^k B^n C_k - f_{mn}^k A_k \eta^n) + \star B_m^* (d\eta^m + f_{nk}^m B^n \eta^k) + \frac{1}{2} \star \eta_k^* f_{mn}^k \eta^m \eta^n - \star C^{*m} f_{mn}^k C_k \eta^n]. \quad (5.3)$$

We follow the conventions of [36] : if $\omega = \frac{1}{p!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \omega_{\mu_1 \dots \mu_p}$, then $\star \omega = \frac{1}{p!(n-p)!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n-p}} \epsilon_{\mu_1 \dots \mu_{n-p}} \omega^{\mu_{n-p+1} \dots \mu_n}$ where $\epsilon^{\mu_1 \dots \mu_n}$ is completely skew symmetric, $\epsilon^{01 \dots n-1} = 1$, and indices are lowered and raised with $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$.

The associated infinitesimal deformation of the master action is

$$S^{(1)} = \int \frac{1}{3!} \beta_{mnk} (B^m B^n B^k + 6 \star A^{*m} B^n \eta^k + 3 \star C^{*m} \eta^n \eta^k). \quad (5.4)$$

In particular, $(S^{(1)}, S^{(1)}) = 0$, so the infinitesimal deformation is by itself a complete deformation, $S^{\text{def}} = S + \mu S^{(1)}$. It plays the role of the topological deformation [52, 53], that has been called “exotic” term in [10]. On the level of the Lie algebra, it amounts to keeping the same inner product but deforming the commutator $[l_m, l_n]^{\text{def}} = f_{mn}^k l_k + \mu \beta_{mnk} l^{*k}$ while keeping all other commutation relations unchanged. That this is legitimate can be checked a posteriori: indeed, the inner product stays invariant because of the complete skew-symmetry of β_{mnk} , while for the Jacobi identity, only $[l_l, [l_m, l_n]^{\text{def}}]^{\text{def}} + \text{cyclic}(l, m, n)$ is non trivial. Evaluating, one finds that this vanishes on account of the cocycle condition since $f_{mn}^k \beta_{klr} + f_{lr}^k \beta_{kmn} + \text{cyclic}(l, m, n)$ is completely skew-symmetric in (l, m, n, r) .

In degree $(0, 3)$, there is no class, because one would need a skew-symmetric three index tensor that is invariant under the coadjoint representation of the Witt algebra. The only candidate is the unique invariant tensor δ_{-101}^{mnk} under the coadjoint representation of $\mathfrak{sl}(2, \mathbb{R})$, which is no longer invariant under transformations generated by l_l with $l \neq -1, 0, 1$. Note that in the case of $\mathfrak{sl}(2, \mathbb{R})$, this is the one that is responsible for the cosmological constant deformation. Whether there are classes in degrees $(1, 2)$ and $(2, 1)$ needs to be investigated.

Let us now turn to the centrally extended case of the Lie algebra $\mathfrak{vir} \ltimes \mathfrak{vir}^*$.

First, note that $H^1(\mathfrak{vir}) = 0 = H^2(\mathfrak{vir})$. Indeed, by denoting by C^a the ghosts associated to the witt algebra and by C^Z the ghost associated to the central element, Chevalley-Eilenberg differential for the Witt algebra is $\gamma = \frac{1}{2} C^a C^b f_{ab}^c \frac{\partial}{\partial C^c}$, while the one for the Virasoro algebra is $\gamma^T = \frac{1}{2} C^a C^b f_{ab}^c \frac{\partial}{\partial C^c} + \omega \frac{\partial}{\partial C^Z}$, where $\omega = \omega_{[ab]} C^a C^b$ is a representative of $H^2(\mathfrak{witt})$, which is 1 dimensional.

For the Virasoro algebra, the cocycle condition in degree 1 for an element $k_a C^a + k C^Z$ implies that $k\omega = -\gamma(k_a C^a)$. This implies in turn $k = 0$ and, since $H^1(\mathfrak{witt}) = 0$, that $k_a C^a$ is γ and also γ^T exact, which gives the result.

The cocycle condition in degree 2 splits depending on whether it involves C^Z or not. The former piece implies that the coefficient of C^Z is a γ cocycle in degree 1; $H^1(\mathfrak{witt}) = 0$ then implies that it is a coboundary, and since there are no coboundaries in degree 1, that it vanishes. The piece independent of the C^Z ghost then has to be a γ^T and thus also a γ cocycle, which implies that it is given by ω up to a γ coboundary $\gamma\eta^1$, where η^1 depends on the C^a alone, so that $\gamma\eta^1 = \gamma^T\eta_1$. But, by construction, $\omega = \gamma^T C^Z$ which proves the result.

A similar reasoning using in addition that ω^2 is a non trivial cohomology class in $H^4(\mathfrak{witt})$ then allows one to show that $H^3(\mathfrak{vir})$ is one dimensional and also described by β . Indeed, for an element $\alpha = k_{[abc]}C^aC^bC^c + k_{[ab]}C^aC^bC^Z$, the cocycle condition $\gamma^T\alpha = 0$ implies $\gamma(k_{[ab]}C^aC^b) = 0$ so that $k_{[ab]}C^aC^b = \gamma(k_aC^a) + k\omega$. This implies that $\alpha = k'_{[abc]}C^aC^bC^c + \gamma^T(k_aC^aC^Z) + k\omega C^Z$, and the cocycle condition becomes $\gamma(k'_{[abc]}C^aC^bC^c) + k\omega^2 = 0$ which implies $k = 0$ and $k'_{[abc]}C^aC^bC^c = k\beta + \gamma(l_{[ab]}C^aC^b) = \beta + \gamma^T(l'_{[ab]}C^aC^b + l_aC^aC^Z)$. Finally, $k\beta = \gamma^T(l'_{[ab]}C^aC^b + l_aC^aC^Z)$ implies $\gamma l_aC^a = 0$ which gives $l_aC^a = 0$ and then $k = 0$, so that β remains non trivial.

6 Dual boundary theory

Following closely the case of $\mathfrak{iso}(2,1)$ treated in [42, 44], the boundary theory when $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2, \mathbb{R})$ is replaced by the Witt or Virasoro algebra is

$$I[\lambda, \alpha] = \frac{k}{2\pi} \int dud\phi \langle \dot{\lambda}\lambda^{-1}, \alpha' \rangle, \quad (6.1)$$

where $\lambda(u, \phi)$ is a map from the cylinder to the $\text{Diff}(S^1)/\text{Virasoro}$ group, $\alpha(u, \phi)$ a map whose value is an associated coadjoint vector, and the inner product $\langle \cdot, \cdot \rangle$ between coadjoint vectors and Lie algebra elements replaces 2Tr in [44]. Note that in that reference, the integrand contains an additional term $-\text{Tr}(\lambda'_{\text{SL}(2, \mathbb{R})}\lambda_{\text{SL}(2, \mathbb{R})}^{-1})^2$ originating from a Gibbons-Hawking type improvement term needed to account for the non trivial asymptotics of the spin connection. Because of the absence of an invariant trace, this term cannot be extended to a $\text{Diff}(S^1)/\text{Virasoro}$ group element. In order to continue to describe an extension of gravity and include the u -dependent solutions discussed at the end of section 4, one should add this terms, with the group element restricted to $\text{SL}(2, \mathbb{R})$. For simplicity, we choose not to do so here and concentrate on the model defined by (6.1).

The equations of motion of the model are

$$(\dot{\lambda}\lambda^{-1})' = 0, \quad D_u^{-}\dot{\lambda}\lambda^{-1}\alpha' = 0. \quad (6.2)$$

The general solution involves a factorized group element $\lambda = \mu(u)\nu(\phi)$, and $\alpha = \mu(\rho(\phi) + \delta(u))\mu^{-1}$. The gauge invariance of the action is $\lambda \rightarrow \nu(u)\lambda$, $\alpha \rightarrow \nu\alpha\nu^{-1}$,

while the global symmetries are $\lambda \rightarrow \lambda \Theta(\phi)^{-1}$, $\alpha \rightarrow \alpha + \lambda \Sigma(\phi) \lambda^{-1}$. The infinitesimal version of the latter are $\delta_\theta \lambda = -\lambda \theta$, $\delta_\theta \alpha = 0$, and $\delta_\sigma \lambda = 0$, $\delta_\sigma \alpha = \lambda \sigma \lambda^{-1}$ with Noether currents

$$\begin{aligned} J_\theta^0 &= \langle \theta, J \rangle, \quad J = -\frac{k}{2\pi} [\lambda^{-1} \alpha' \lambda], \quad J_\theta^1 = 0, \\ P_\sigma^0 &= \langle \sigma, P \rangle, \quad P = \frac{k}{2\pi} \lambda^{-1} \lambda', \quad P_\sigma^1 = 0. \end{aligned} \quad (6.3)$$

In terms of generators, $J_m = \langle l_m, J \rangle$, $\zeta = \langle Z, J \rangle$, and $P^m = \langle l^{*m}, P \rangle$, $\zeta^* = \langle Z^*, P \rangle$, one then reads off the current algebra from the structure constants and the inner product,

$$\begin{aligned} \{P^m(\phi), P^n(\phi')\}^* &= 0 = \{P^m(\phi), \zeta(\phi')\}^* = \{P^m(\phi), \zeta^*(\phi')\}^*, \\ \{\zeta(\phi), \zeta^*(\phi')\}^* &= -\frac{k}{2\pi} \partial_\phi \delta(\phi - \phi'), \quad \{\zeta(\phi), J_m(\phi')\}^* = 0, \\ \{J_m(\phi), \zeta^*(\phi')\}^* &= -\frac{1}{12} m(m^2 - 1) P^{-m}(\phi) \delta(\phi - \phi'), \\ \{J_m(\phi), P^n(\phi')\}^* &= (n - 2m) P^{n-m}(\phi) \delta(\phi - \phi') - \frac{k}{2\pi} \delta_m^n \partial_\phi \delta(\phi - \phi'), \\ \{J_m(\phi), J_n(\phi')\}^* &= [(m - n) J_{m+n}(\phi) + \frac{\zeta(\phi)}{12} \delta_{m+n}^0 m(m^2 - 1)] \delta(\phi - \phi'). \end{aligned} \quad (6.4)$$

The natural combination that is available for a classical Sugawara construction is

$$\mathcal{P} \approx -\frac{2\pi}{k} (J_m P^m + \zeta \zeta^*), \quad (6.5)$$

and

$$\{\mathcal{P}(\phi), \mathcal{P}(\phi')\}^* = (\mathcal{P}(\phi) + \mathcal{P}(\phi')) \partial_\phi \delta(\phi - \phi'), \quad (6.6)$$

or, in terms of modes $\mathcal{J}_m = \int_0^{2\pi} d\phi e^{im\phi} \mathcal{P}$,

$$i\{\mathcal{J}_m, \mathcal{J}_n\} = (m - n) \mathcal{J}_{m+n}, \quad (6.7)$$

which expresses conformal invariance of the model.

Acknowledgements

This work is supported by two FNRS/MINCYT collaboration agreements. The work of G.B. and G.G. is partially funded by the Fund for Scientific Research-FNRS Belgium (convention FRFC PDR T.1025.14 and convention IISN 4.4503.15) and by “Communauté française de Belgique - Actions de Recherche Concertées”. The work of G.G. is also supported by a donation from the Solvay family.

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